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TECHNICAL NOTE

Graphic Examples of a Logical Nonlinear Strength Criterion for Intact Rock

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List of Symbols

c	Apparent cohesion (MPa)
c_0	Apparent cohesion when $\sigma_n = 0$ (MPa)
c'	Derivative of c with respect to σ_n
m	Material parameter in Hoek–Brown criterion
σ_1	Major principal stress (MPa)
σ_3	Minor principal stress (MPa)
σ_c	Uniaxial compressive strength (MPa)
σ_n	Normal stress on potential shearing plane (MPa)
σ_t	Uniaxial tensile strength (MPa)
ϕ	Apparent internal friction angle
ϕ_0	Apparent internal friction angle when $\sigma_n = 0$
ϕ'	Derivative of ϕ with respect to σ_n
ϕ^*	Inclination angle of the tangent slope of the strength envelope

1 Introduction

It is well-known that the dependence of shear strength of rocks on the normal stress acting on the potential shear plane is nonlinear, and that rocks are weaker than predicted by the traditional linear Coulomb shear strength criterion, especially when the normal stress is large. There are many nonlinear shear strength criteria in the literature, such as the Barton criterion (Barton 1976, 2006, 2013) and Hoek–Brown criterion (Hoek and Brown 1980a, b, 1988). Barton (2013) presented a summary of the nonlinear shear strengths for intact rocks, fractured rocks, jointed rocks and rockfills. As with most of these laboratory-tested geologic media, shear strength envelopes in the τ–σ_n plane are convex in relation to the normal compression stress axis, where τ is the shear strength and σ_n is the normal stress on the shear plane.

Based on many high-pressure triaxial experiments in the literature, including Mogi (1966) and Byerlee (1968), Barton (1976) proposed a critical state concept for rocks. At the critical state, the tangent of the shear strength envelope approaches horizontal in the τ–σ_n plane (Barton 2006, 2013). For confining pressure greater than the critical value, which is close to the unconfined compressive strength, the shear strength will not increase anymore. The value of peak shear strength is half of the normal compressive stress. The proposed non-linearity and reasons for such opinions are indicated in Fig. 1.

Singh et al. (2011) incorporated the critical state concept into a modified triaxial strength criterion for intact rocks expressed in a relation of σ_1–σ_3 and σ_3 and, with a large set of data, showed that the critical confining pressure is approximately equal to the UCS, which agrees with Barton's suggestion from 1976. Recently, Shen et al. (2018) further investigated the critical state concept for rocks and proposed a simple modified nonlinear shear strength criterion, which is of the classical Coulomb criterion form, but with the cohesion and internal frictional angle depending on the normal stress. It covers both compression and tensile regimes of failure. In this technical note, we first show graphs of the nonlinear shear strength using four sets of parameters, and then develop an approximate conversion of the shear strength criterion proposed by Shen et al. (2018) into one in terms of $\sigma_1 \sim \sigma_3$. It is noted that the critical state concept for rock failure has also been employed by others, for example, Carroll (1991) and Baud et al. (2006), as reviewed by Wong and Baud (2012), with a different form from the one we proposed.

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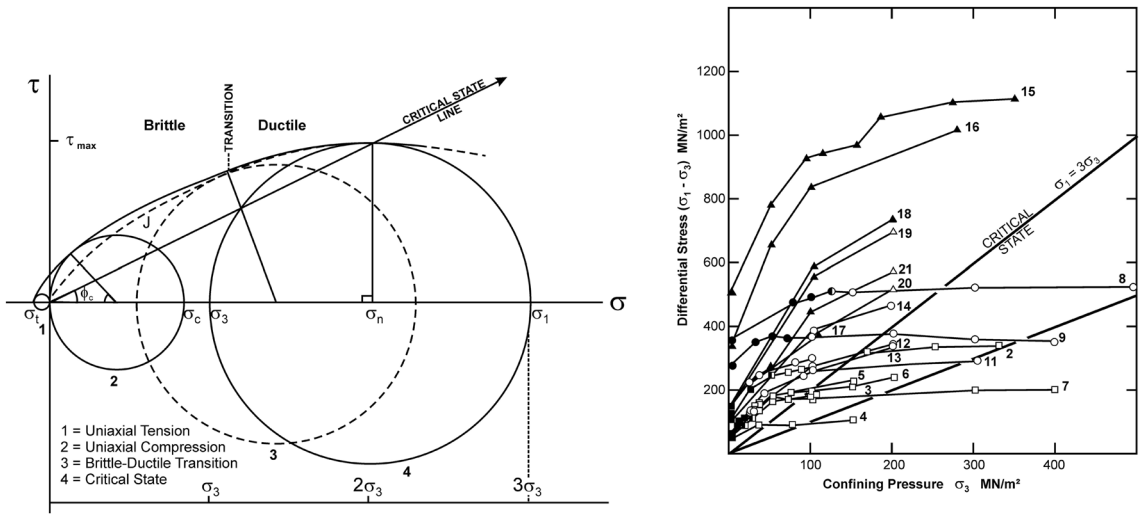

Fig. 1 The suggested non-linearity and critical state suggestion of Barton (1976) which was based on sets of high-pressure triaxial test data as presented by Mogi (1966). One set is shown here

2 Modified Approximate Nonlinear Strength Criterion with Critical State in Terms of $\tau \sim \sigma_n$

Shen et al. (2018) incorporated the critical state concept for rocks and proposed a modified nonlinear shear strength criterion that has the same form as the classical Coulomb criterion in terms of τ and σ_n:

$$\tau = \sigma_n \tan\phi + c \tag{1}$$

for both compressive and tensile regimes, but with both ϕ and c being functions of the normal stress σ_n. In the compression region ($\sigma_n > 0$), the functions have the forms:

$$\phi = \phi_0\left(1 - \sqrt{\frac{\sigma_n}{2\sigma_c}}\right) \tag{2}$$

$$c = c_0 + (\sigma_c - c_0)\frac{\sigma_n}{2\sigma_c} \tag{3}$$

and in the tension region ($\sigma_n < 0$), they are:

$$\phi = \phi_0 - (45^\circ - \phi_0)\frac{\sigma_n}{\sigma_t} \tag{4}$$

$$c = c_0 + (\sigma_t - c_0)\left(\frac{\sigma_n}{\sigma_t}\right)^2 \tag{5}$$

with σ_t, σ_c, ϕ_0 and c_0 being material parameters. At $\sigma_n = -\sigma_t$, the shear strength $\tau = 0$, thus σ_t is the unconfined tensile strength (UTS) of the material. The envelope approximately touches the Mohr circle corresponding to the stress state of $\sigma_1 = \sigma_c$, $\sigma_3 = 0$. So σ_c can be taken as the unconfined compression strength (UCS) of the material. At $\sigma_n = 0$, $\tau = c_0$ from both the compression and tension sides. Hence, the two parts are continuously joined at $\sigma_n = 0$ and c_0 is the cohesion of the material. But the tangent slopes at $\sigma_n = 0$ of the two parts are different. From the tension region, it is $\tan\phi_0$, while from the compression region, it is $\tan\phi_0 + \frac{1}{2} - \frac{c_0}{2\sigma_c}$. Generally, $c_0 \neq \sigma_c$, so ϕ_0 is not the real internal friction angle at $\sigma_n = 0$. Since ϕ_0 and c_0 are closely related to the nonlinear strength around $\sigma_n = 0$ and the nonlinear strength here is very close to the classical linear Coulomb criterion, for simplicity, the classical form of relation among the three parameters σ_c, ϕ_0 and c_0 is employed:

$$\sigma_c = \frac{2c_0\cos\phi_0}{1 - \sin\phi_0} \text{ or } c_0 = \frac{1 - \sin\phi_0}{2\cos\phi_0}\sigma_c. \tag{6}$$

It can also be seen that at $\sigma_n = 2\sigma_c$, the expressions in Eqs. (1)–(3) give $\tau = \sigma_c$. The tangent of the strength envelope reaches horizontal around $\sigma_n = 2\sigma_c$. These are the apparent conditions for critical state, see Barton (1976) and Fig. 1. So the simple modified strength criterion reflects the critical state concept.

Figure 2 shows four strength envelopes for four sets of material parameters, together with related Mohr circles. The material parameters are listed in Table 1, in which c_0 is calculated with the second one of Eq. (6). These can serve as guides for engineering applications.

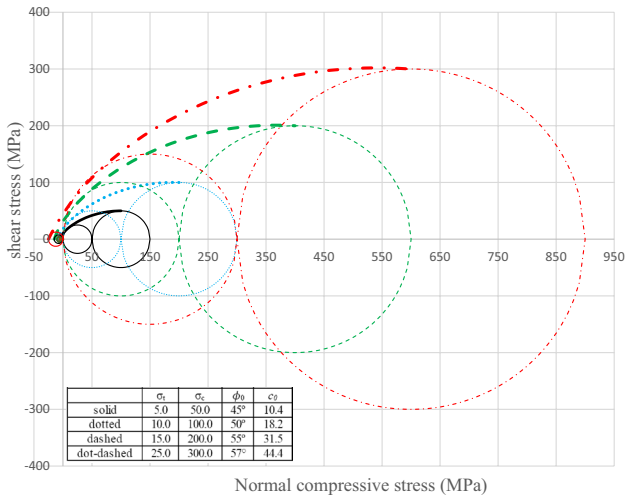

Fig. 2 Four graphs illustrating the modified nonlinear strength criterion, with the relevant Mohr circles for four example sets of material parameters

Table 1 Values of the example strength parameters used in Figs. 2 and 4

	σ_t (MPa)	σ_c (MPa)	ϕ_0	c_0 (MPa)
Set 1 (black: solid)	5.0	50.0	45°	10.4
Set 2 (blue: dotted)	10.0	100.0	50°	18.2
Set 3 (green: dashed)	15.0	200.0	55°	31.5
Set 4 (red: dot-dashed)	25.0	300.0	57°	44.4

3 Strength Criterion with Critical State in Terms of $\sigma_1 \sim \sigma_3$

In some situations, it is better to have the strength criterion expressed in terms of $\sigma_1 \sim \sigma_3$, as for instance that of the Hoek–Brown criterion (Hoek and Brown 1988). Here we transfer the strength criterion given by Eqs. (1)–(5) into equivalent but approximate $\sigma_1 \sim \sigma_3$ terms.

At any point on the shear strength envelope, there is a corresponding Mohr circle in the $\tau \sim \sigma_n$ plane which is tangentially touching the envelope at the point. The stress state corresponding to the touching point is given by

$$\sigma_n = \frac{\sigma_1 + \sigma_3}{2} - \frac{\sigma_1 - \sigma_3}{2} \sin \phi^*, \tau = \frac{\sigma_1 - \sigma_3}{2} \cos \phi^* \tag{7}$$

(see Fig. 3). Here σ_1 and σ_3 are the major and minor principal stresses and ϕ^* is the inclination angle of the tangent slope of the strength envelope at the point, in relation to the normal stress σ_n axis. It is noted that due to the dependence of ϕ and c on the normal stress σ_n, ϕ^* is generally different from ϕ.

Since this point is on the strength envelope, we have

$$\frac{\sigma_1 - \sigma_3}{2} \cos \phi^* = \left\{ \frac{\sigma_1 + \sigma_3}{2} - \frac{\sigma_1 - \sigma_3}{2} \sin \phi^* \right\} \tan \phi + c.$$

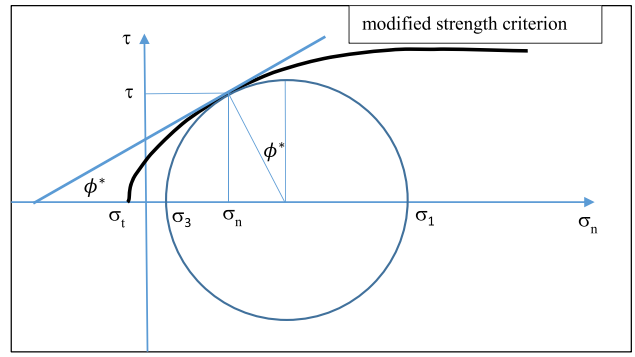

Fig. 3 Mohr circle for stress state of points on the modified strength criterion envelope

Since $\tan \phi = \sin \phi / \cos \phi$, this can be expressed as

$$(\sigma_1 - \sigma_3)(\cos \phi^* \cos \phi + \sin \phi^* \sin \phi) = (\sigma_1 + \sigma_3) \sin \phi + 2c \cos \phi. \tag{8}$$

Through the expression (7), ϕ and c depend on σ_1, σ_3 and angle ϕ^*. Thus, (8) is a nonlinear relation between σ_1 and σ_3 with an extra variable ϕ^*. We need to express ϕ^* in terms of σ_1 and σ_3 to convert (8) into a pure relation between σ_1 and σ_3.

The tangent angle ϕ^* should satisfy

$$\tan \phi^* = \frac{\partial \tau}{\partial \sigma_n} = \tan \phi + \sigma_n \frac{\phi'}{\cos^2 \phi} + c' \tag{9}$$

with ϕ' and c' being derivatives of ϕ and c, respectively, with respect to the normal stress σ_n. Through the expression (7), Eq. (9) gives a relation between ϕ^*, σ_1 and σ_3. However, it is hard, if not impossible, to express ϕ^* in terms of σ_1 and σ_3 from this relation. Thus, it would be very hard, maybe even impossible, to express the nonlinear strength criterion (8) analytically in $\sigma_1 \sim \sigma_3$ format. We will solve (8) and (9) for σ_1 and ϕ^* numerically for given values of σ_3 with specified values of the material parameters and express the strength criterion graphically in the $\sigma_1 \sim \sigma_3$ plane. For reasons of clarity, we will plot four curves corresponding to the four curves given in Fig. 2.

Some particular points can be identified and are needed to guide the numerical solution.

1. The point $(-\sigma_t, 0)$ in the $\tau \sim \sigma_n$ plane corresponds to $(-\sigma_t, -\sigma_t)$ in the $\sigma_1 \sim \sigma_3$ plane.
2. The separation point of tensile and compression regimes, $(0, c_0)$, satisfies

$$\frac{\sigma_1 + \sigma_3}{2} - \frac{\sigma_1 - \sigma_3}{2} \sin \phi_0^* = 0, \frac{\sigma_1 - \sigma_3}{2} \cos \phi_0^* = c_0.$$

Here ϕ_0^* is the tangential slope angle of the envelope at the point $(\sigma_n = 0)$ in the $\tau \sim \sigma_n$ plane. The above analysis shows that ϕ_0^* has a slightly different value in the ten-

sile region from that in the compression region. Taking $\phi_0^* = \phi_0$ from the tensile region, this point corresponds to $[-c_0(1-\sin\phi_0)/\cos\phi_0, c_0(1+\sin\phi_0)/\cos\phi_0]$ in the $\sigma_1 \sim \sigma_3$ plane.

3. The state of stress representing uniaxial compression strength is $(0, \sigma_c)$ in the $\sigma_1 \sim \sigma_3$ plane.
4. The state of stress at the critical state is $(\sigma_c, 3\sigma_c)$ in the $\sigma_1 \sim \sigma_3$ plane, which was predicted by Barton (1976).

Since the equations are highly nonlinear, the initial estimates of the solutions for the iteration are very important. It is noted that the Newton–Raphson iteration for σ_1 and ϕ^* has convergence difficulty for the points around $\sigma_n = 0$, particularly for points on the compression side. This is due to the change of the sign of the normal stress around the point, which leads to using different equations in the calculations. We were not able to obtain a satisfactory solution of the σ_1 and ϕ^* for the first point next to $\sigma_n = 0$, based on 60 calculation points on the compression side for the third and fourth cases of $\sigma_t = 15$ and 25 MPa (see Table 1), nor for the first two points for the first and second cases of $\sigma_t = 5$ and 10 MPa. These points were therefore ignored, and the next point and the tensile part were linked to produce the curve. The four strength curves in the $\sigma_1 \sim \sigma_3$ plane corresponding to the four curves in the $\tau \sim \sigma_n$ plane in Fig. 2 are shown in Fig. 4. The curves are all ended at critical state $\sigma_3 = \sigma_c$.

It has been noted that in the $\tau \sim \sigma_n$ format, the strength curves are not smooth at the point between the tensile and compressive regions. It is expected that the curves in the $\sigma_1 \sim \sigma_3$ plane are also not smooth at the corresponding point, which is at $(-c_0(1-\sin\phi_0)/\cos\phi_0, c_0(1+\sin\phi_0)/\cos\phi_0)$. It is acceptable to have non-smooth strength curves at some isolated points, for example, the well-known Tresca yield surface is not smooth. From Fig. 4, it can be seen that the non-smoothness is not severe for the four cases.

To further show the applicability of the modified criterion, here we compare the curve for the case of $\sigma_t = 5$ MPa and $\sigma_c = 50$ MPa with that of Hoek–Brown nonlinear criterion in Fig. 5. The parameter m of Hoek–Brown criterion for intact rock, uniaxial tensile strength and unconfined compressive strength has the relationship: $m = \sigma_c/\sigma_t - \sigma_t/\sigma_c$, which is derived from the relation (6) given in Hoek and Brown (1988) (with $s = 1$ for intact rock). For this case, $m = 9.9$ and the non-smooth point is at $(-4.31, 25.11)$. Both curves go through the point corresponding to unconfined compressive strength $(0, \sigma_c = 50)$ point. It can be seen that in the region $\sigma_3 < 0$, the modified criterion predicts slightly higher strength, while in the region $\sigma_3 > 0$, Hoek–Brown criterion predicts higher strength.

4 Discussion and Conclusions

1. The extreme non-linearity of a wide range of high-pressure triaxially tested dry silicate and dry carbonate rocks, as presented by Mogi (1966) was the original source of a suggested critical state for rock given by Barton (1976). This was illustrated in Fig. 1. Quantitatively, at the critical state, the major principal stress (σ_1) is about three times the minor one (σ_3). Significantly, the minor principal stress (or critical confining pressure) is close to or equal to the UCS, based on an extensive review of more than 1000 triaxial tests by Singh et al. (2011).
2. The value of peak shear strength is therefore seen to be approximately half of the normal compressive stress at the assumed critical state. Mohr circle geometry (Fig. 1) indicates that the UCS circle diameter is approximately equal to the radius of the critical state circle. Numerically speaking, the maximum possible shear strength

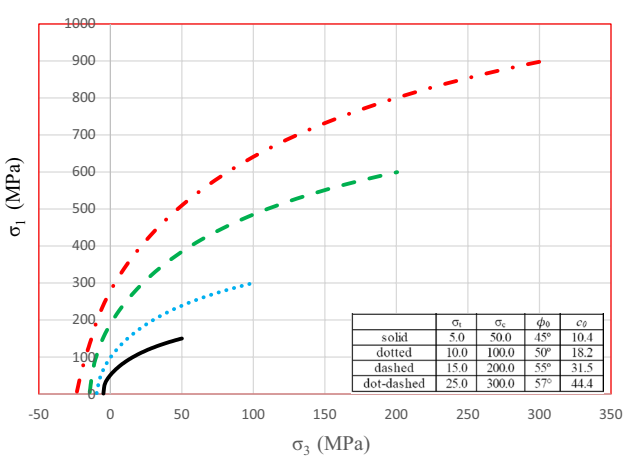

Fig. 4 Approximate strength curves in $\sigma_1 \sim \sigma_3$ format, for the four cases shown in Fig. 2

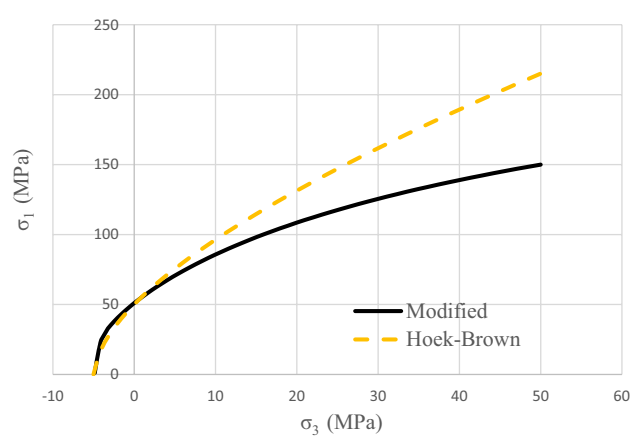

Fig. 5 Comparison of the modified criterion with Hoek–Brown criterion for the case of $\sigma_t = 5$ MPa and $\sigma_c = 50$ MPa

($\tau_{critical}$) therefore has the same approximate magnitude as UCS.

3. In this technical note, we have presented strongly nonlinear strength envelopes in terms of shear and normal stress ($\tau \sim \sigma_n$) and a more approximate solution in terms of ($\sigma_1 \sim \sigma_3$), with four sets of familiar input data in each case, and with strength evaluated up to the critical state. Comparison with the Hoek–Brown criterion for one case was shown.
4. Singh et al. (2011) also investigated the effect of the intermediate principal stress on strength, using polyaxial test data. They found that the critical state condition was satisfied by both the minimum and intermediate principal stresses. Singh and Singh (2012) later employed the same form of strength criterion for jointed rock masses.

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Compliance with Ethical Standards

Conflict of Interest The authors know of no conflict of interest concerning publication of this technical note.

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